

Regular completions of \mathbb{Z}^n -free groups

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Abstract

In the present paper we continue studying regular free group actions on \mathbb{Z}^n -trees. We show that every finitely generated \mathbb{Z}^n -free group G can be embedded into a finitely generated \mathbb{Z}^n -free group H acting regularly on the underlying \mathbb{Z}^n -tree (we call H a *regular \mathbb{Z}^n -completion* of G) so that the action of G is preserved. Moreover, if G is effectively represented as a group of \mathbb{Z}^n -words then the construction of H is effective and H is effectively represented as a group of \mathbb{Z}^n -words.

1 Introduction

The theory of Λ -trees (where $\Lambda = \mathbb{R}$) has its origins in the papers by I. Chiswell [4] and J. Tits [32]. In particular, in the latter paper the definition of \mathbb{R} -tree was given, while the former one established the fundamental connection between group actions on trees and length functions on groups introduced in 1963 by R. Lyndon (see [23]). Length functions were introduced in an attempt to axiomatize cancellation arguments in free groups as well as free products with amalgamation and HNN extensions, and to generalize them to a wider class of groups. The main idea was to measure the amount of cancellation in passing to the reduced form of a product of reduced words in a free group and free constructions, and it turned out that the cancellation process could be described by rather simple axioms. Using simple combinatorial techniques Lyndon described groups with *free* \mathbb{Z} -valued length functions (such length functions correspond to actions on simplicial trees without fixed points).

Later, in their very influential paper [24] J. Morgan and P. Shalen linked group actions on \mathbb{R} -trees with topology and generalized parts of Thurston's Geometrization Theorem. Next, they introduced Λ -trees for an arbitrary ordered abelian group Λ and the general form of Chiswell's construction. Thus, it became clear that abstract length functions with values in Λ and group actions on Λ -trees are just two equivalent approaches to the same realm of group theory questions. The unified theory was further developed in the important paper by R. Alperin and H. Bass [1], where authors state a fundamental problem in the theory of group actions on Λ -trees: find the group theoretic information carried by a Λ -tree action (analogous to Bass-Serre theory), in particular, describe finitely generated groups acting freely on Λ -trees (Λ -free groups). One of the main breakthroughs in this direction is Rips' Theorem, that describes

finitely generated \mathbb{R} -free groups (see [10, 3]). The structure of finitely generated \mathbb{Z}^n -free can be deduced from [2] using Lyndon's results (see [23]) and inductive argument on n , while the structure of \mathbb{R}^n -free groups was clarified in [12] using ideas of [3] and again induction on n .

Introduction of infinite Λ -words was one of the major recent developments in the theory of Λ -free groups. In [25] A. Myasnikov, V. Remeslennikov and D. Serbin showed that groups admitting faithful representations by Λ -words act freely on Λ -trees, while Chiswell proved the converse [6]. This gives another equivalent approach to the whole theory so that one can replace the axiomatic viewpoint of length functions along with many geometric arguments coming from Λ -trees by combinatorics of Λ -words. In particular, this approach allows one to naturally generalize powerful techniques as Nielsen's method, Stallings' graph approach to subgroups, and Makanin-Razborov type of elimination processes from free groups to Λ -free groups (see [25, 26, 16, 17, 18, 19, 9, 21, 20, 27, 28, 30]). In the case when Λ is equal to either \mathbb{Z}^n or \mathbb{Z}^∞ all these techniques are effective, so, many algorithmic problems for \mathbb{Z}^n -free groups become decidable.

While studying Λ -free groups it becomes evident that it is necessary to introduce some natural restrictions on the action which could significantly simplify many arguments. Thus, given a group G acting on a Λ -tree Γ , we say that the action is *regular with respect to* $x \in \Gamma$ (see [19] for details) if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$. In fact, the definition above does not depend on x and there exist equivalent formulations for length functions and Λ -words (see [29, 25]). Roughly speaking, regularity of action implies that all branch-points of Γ belong to the same G -orbit and it tells a lot about the structure of G in the case of free actions (see [20, 19]). Now, given a finitely generated group G acting freely on a Λ -tree Γ , several natural questions arise:

- When does G admit a regular action on Γ ?
- Is it possible to change the action of G on Γ in order to make it regular?
- Is it possible to embed G into a finitely generated Λ -free group H which admits a regular action? Can one do it in an equivariant manner (in this case we call such H a *regular Λ -completion* of G)?

In particular, the last question has a positive answer (see [23, 13]) in the case when $\Lambda = \mathbb{Z}$ with H being finitely generated (the construction of H is effective in this case). The general case is approached in [7], where the group H is constructed but it is almost never finitely generated (even if G is finitely generated \mathbb{Z}^n -free).

In this paper we answer the third question above affirmatively and show that a \mathbb{Z}^n -completion of G can be found effectively if one starts with an effective representation of G by infinite words. In particular, the following theorem is proved.

Theorem 4. *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$, where X is arbitrary. Then there exists a finite alphabet Y and an embedding $\phi :$*

$G \rightarrow H$, where H is a finitely generated subgroup of $CDR(\mathbb{Z}^n, Y)$ with a regular length function, such that $|g|_G = |\phi(g)|_H$ for every $g \in G$. Moreover, if G has effective an hierarchy over X then H an effective hierarchy over Y .

2 Preliminaries

Here we introduce basics of the theory of Λ -trees (all the details can be found in [1] and [5]), Lyndon length functions (see [23, 4]) and infinite words (see [25]).

2.1 Λ -trees

Let Λ be an ordered abelian group. A Λ -metric space is a pair (X, d) , where $d : X \times X \rightarrow \Lambda$ is a function such that for all $x, y, z \in X$:

- (M1) $d(x, y) \geq 0$,
- (M2) $d(x, y) = 0$ if and only if $x = y$,
- (M3) $d(x, y) = d(y, x)$,
- (M4) $d(x, y) \leq d(x, z) + d(y, z)$,

in other words, the function d satisfies metric axioms with \mathbb{R} replaced by Λ .

The simplest example of a Λ -metric space is Λ itself with the metric $d(a, b) = |a - b|$, for all $a, b \in \Lambda$. For every $a, b \in \Lambda$ such that $a \leq b$ define $[a, b]_\Lambda = \{x \in \Lambda \mid a \leq x \leq b\}$.

As usual, if (X, d) and (X', d') are Λ -metric spaces, an *isometry* from (X, d) to (X', d') is a mapping $f : X \rightarrow X'$ such that $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$. Thus, for $x, y \in X$, a *segment* $[x, y]$ in a X is the image of an isometry $\alpha : [a, b]_\Lambda \rightarrow X$ for some $a, b \in \Lambda$ such that $\alpha(a) = x$ and $\alpha(b) = y$ (observe that $[x, y]$ is not unique in general). We call a Λ -metric space (X, d) *geodesic* if at least one segment $[x, y]$ exists for all $x, y \in X$, and (X, d) is *geodesically linear* if $[x, y]$ is unique for all $x, y \in X$.

Now, a Λ -metric space (X, d) is called a Λ -tree if

- (T1) (X, d) is geodesic,
- (T2) if two segments of (X, p) intersect in a single point that is an endpoint of both, then their union is a segment,
- (T3) the intersection of two segments with a common endpoint is also a segment.

Example 1. Λ together with the metric $d(a, b) = |a - b|$ is a Λ -tree.

Example 2. A \mathbb{Z} -metric space (X, d) is a \mathbb{Z} -tree if and only if there is a simplicial tree Γ such that $X = V(\Gamma)$ and d is the path metric of Γ .

We say that a group G *acts on a Λ -tree X* if any element $g \in G$ defines an isometry $g : X \rightarrow X$. G acts on X *freely* and *without inversions* if no non-trivial $g \in G$ stabilizes a segment in X (a segment can be degenerate). In this case we say that G is Λ -free. The action is *regular with respect to $x \in X$* if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$ (see [21]).

Given an action of G on a Λ -tree (X, d) , for a point $x \in X$ one can define a function $l_x : G \rightarrow \Lambda$ by $l_x(g) = d(x, gx)$. Such a function is called a *based length function on G* and it is easy to check that l_x satisfies the axioms

- (L1) $\forall g \in G : l_x(g) \geq 0$ and $l_x(1) = 0$,
- (L2) $\forall g \in G : l_x(g) = l_x(g^{-1})$,
- (L3) $\forall g, f, h \in G : c_x(g, f) > c_x(g, h) \rightarrow c_x(g, h) = c_x(f, h)$,
where $c_x(g, f) = \frac{1}{2}(l_x(g) + l_x(f) - l_x(g^{-1}f))$.

Moreover, if G is Λ -free then

- (L4) $\forall g \in G : l_x(g^2) > l_x(g)$,

and regularity of the action implies the axiom

- (R) $\forall g, h \in G, \exists u, g_1, h_1 \in G :$

$$g = f \circ g_1 \text{ \& } h = f \circ h_1 \text{ \& } l_x(f) = c_x(g, h),$$

where $vw = v \circ w$ means that $l_x(vw) = l_x(v) + l_x(w)$.

Now, one can consider an abstract function $l : G \rightarrow \Lambda$ with the axioms (L1)–(L3), it is called a *Lyndon length function* on G . One can show that for such a function l there exists a Λ -tree (X, d) and a point $x \in X$ such that $l = l_x$ provided $c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)) \in \Lambda$ for all $f, g \in G$ (see, for example [5, Theorem 2.4.6]). l is called *free* if it satisfies (L4) and it is called *regular* if it satisfies (R).

In this paper we are mostly interested in groups with regular free Lyndon length functions and here are examples.

Example 3. Let $F = F(X)$ be a free group on X . The length function

$$|\cdot| : F \rightarrow \mathbb{Z},$$

where $|f|$ is a natural length of $f \in F$ as a finite word, is regular since the common initial subword of any two elements of F always exists and belongs to F .

Example 4. In [25] it was proved that Lyndon's free $\mathbb{Z}[t]$ -group has a regular free length function with values in $\mathbb{Z}[t]$.

Example 5. [19] Let $F = F(X)$ be a free group on X . Consider an HNN-extension

$$G = \langle F, s \mid u^s = v \rangle,$$

where $u, v \in F$ are such that $|u| = |v|$ and u is not conjugate to v^{-1} . Then there is a regular free length function $l : G \rightarrow \mathbb{Z}^2$ which extends the natural integer-valued length function on F .

For more involved examples, we refer the reader to [19].

2.2 Infinite words

Let Λ be an ordered abelian group (we refer the reader to the books [11] and [22] regarding the general theory of ordered abelian groups). Λ is called *discretely ordered* if it has a minimal positive element. Let us fix a discretely ordered Λ for the rest of this subsection. Hence, with a slight abuse of notation we denote the minimal positive element of Λ by 1 and the segment $[a, b]_\Lambda$ for $a, b \in \Lambda$ by $[a, b]$.

Now, following [25], given a set $X = \{x_i \mid i \in I\}$, we put $X^{-1} = \{x_i^{-1} \mid i \in I\}$, $X^\pm = X \cup X^{-1}$, and define a Λ -word as a function of the type

$$w : [1, \alpha_w] \rightarrow X^\pm,$$

where $\alpha_w \in \Lambda$, $\alpha_w \geq 0$. The element α_w is called the *length* $|w|$ of w .

In particular, \mathbb{Z} -words are finite words in ordinary sense.

Below we refer to Λ -words as *infinite words* usually omitting Λ whenever it does not produce any ambiguity.

By $W(\Lambda, X)$ we denote the set of all infinite words. Observe, that $W(\Lambda, X)$ contains an empty word which we denote by ε . Operations of concatenation and inversion are defined on $W(\Lambda, X)$ in the usual way (see [25]).

An infinite word w is *reduced* if it does not contain xx^{-1} , $x \in X^\pm$ as a subword and we denote by $R(\Lambda, X)$ the set of all reduced infinite words. Clearly, $\varepsilon \in R(\Lambda, X)$. If the concatenation uv of two reduced infinite words u and v is also reduced then we write $uv = u \circ v$.

For $u \in W(\Lambda, X)$ and $\beta \in [1, |u|]$ by u_β we denote the restriction of u on $[1, \beta]$. If $u \in R(\Lambda, X)$ and $\beta \in [1, |u|]$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined \tilde{u}_β .

An element $\text{com}(u, v) \in R(\Lambda, X)$ is called the *(longest) common initial segment* of reduced infinite words u and v if

$$u = \text{com}(u, v) \circ \tilde{u}, \quad v = \text{com}(u, v) \circ \tilde{v}$$

for some (uniquely defined) infinite words \tilde{u}, \tilde{v} such that $\tilde{u}(1) \neq \tilde{v}(1)$. Note that $\text{com}(u, v)$ does not always exist.

Now, let $u, v \in R(\Lambda, X)$. If $\text{com}(u^{-1}, v)$ exists then

$$u^{-1} = \text{com}(u^{-1}, v) \circ \tilde{u}, \quad v = \text{com}(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined \tilde{u} and \tilde{v} . In this event put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(\Lambda, X)$.

An element $v \in R(\Lambda, X)$ is termed *cyclically reduced* if $v(1)^{-1} \neq v(|v|)$. We say that an element $v \in R(\Lambda, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(\Lambda, X)$ and u is cyclically reduced. Observe that a cyclic decomposition is unique (whenever it exists). We denote by $CDR(\Lambda, X)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

Now we consider *subgroups of $CDR(\Lambda, X)$* , that is, subsets of $CDR(\Lambda, X)$ closed with respect to “ $*$ ” and inversion of infinite words.

Theorem 1. [25] *Any subgroup G of $CDR(\Lambda, X)$ is a group with a free Lyndon length function $|\cdot| : G \rightarrow \Lambda$, where $|g|$ is the length of g viewed as an element of $CDR(\Lambda, X)$.*

The converse is also true.

Theorem 2. [6] *Let G have a free Lyndon length function $l : G \rightarrow \Lambda$, then there exists an embedding $\phi : G \rightarrow CDR(\Lambda, X)$ such that, $|\phi(g)| = l(g)$ for any $g \in G$.*

Moreover, it was shown in [15] that the embedding ϕ in Theorem 2 preserves regularity. Observe that regularity of the length function $|\cdot|$ on a subgroup H of $CDR(\Lambda, X)$ means that $\text{com}(g, h) \in H$ for all $g, h \in H$.

Thus, Λ -free groups are precisely groups with free Λ -valued Lyndon length functions, which are precisely subgroups of $CDR(\Lambda, X)$ for an appropriate X .

2.3 Universal trees

Let G be a subgroup of $CDR(\Lambda, X)$ for some discretely ordered abelian group Λ and a set X .

Briefly recall (see [21] for details) how one can construct a universal Λ -tree Γ_G for G . Every element $g \in G$ is a function

$$g : [1, |g|] \rightarrow X^\pm,$$

with the domain $[1, |g|]$ which is a closed segment in Λ . Since Λ can be viewed as a Λ -metric space then $[1, |g|]$ is a geodesic connecting 1 and $|g|$, and every $\alpha \in [1, |g|]$ can be viewed as a pair (α, g) . Let

$$S_G = \{(\alpha, g) \mid g \in G, \alpha \in [0, |g|]\}.$$

Since for every $f, g \in G$ the word $\text{com}(f, g)$ is defined, one can introduce an equivalence relation on S_G as follows: $(\alpha, f) \sim (\beta, g)$ if and only if $\alpha = \beta \in$

$[0, c(f, g)]$. Now, let $\Gamma_G = S_G / \sim$ and $\varepsilon = \langle 0, 1 \rangle$, where $\langle \alpha, f \rangle$ is the equivalence class of (α, f) . It was shown in [21] that Γ_G is a Λ -tree with a designated vertex ε and a metric $d : \Gamma_G \times \Gamma_G \rightarrow \Lambda$, on which G acts by isometries so that for every $g \in G$ the distance $d(\varepsilon, g \cdot \varepsilon)$ is exactly $|g|$. Moreover, Γ_G is equipped with the labeling function $\xi : (\Gamma_G - \{\varepsilon\}) \rightarrow X^\pm$, where $\xi(v) = g(\alpha)$ if $v = \langle \alpha, g \rangle$.

It is easy to see that the labeling ξ is not equivariant, that is, $\xi(v) \neq \xi(g \cdot v)$ in general (even if both v and $g \cdot v$ are in $\Gamma_G - \{\varepsilon\}$, which is not stable under the action of G). In the present paper we are going to introduce another labeling function for Γ_G defined not on vertices but on “edges”, stable under the action of G . With this new labeling Γ_G becomes an extremely useful combinatorial object in the case $\Lambda = \mathbb{Z}^n$, but in general such a labeling can be defined for every discretely ordered Λ .

First of all, for every $v_0, v_1 \in \Gamma_G$ such that $d(v_0, v_1) = 1$ we call the ordered pair (v_0, v_1) the *edge* from v_0 to v_1 . Here, if $e = (v_0, v_1)$ then denote $v_0 = o(e)$, $v_1 = t(e)$ which are respectively the *origin* and *terminus* of e . Now, if the vertex $v_1 \in \Gamma_G - \{\varepsilon\}$ is fixed then, since Γ_G is a Λ -tree, there is exactly one point v_0 such that $d(\varepsilon, v_1) = d(\varepsilon, v_0) + 1$. Hence, there exists a natural orientation, with respect to ε , of edges in Γ_G , where an edge (v_0, v_1) is *positive* if $d(\varepsilon, v_1) = d(\varepsilon, v_0) + 1$, and *negative* otherwise. Denote by $E(\Gamma_G)$ the set of edges in Γ_G . If $e \in E(\Gamma_G)$ and $e = (v_0, v_1)$ then the pair (v_1, v_0) is also an edge and denote $e^{-1} = (v_1, v_0)$. Obviously, $o(e) = t(e^{-1})$. Because of the orientation, we have a natural splitting

$$E(\Gamma_G) = E(\Gamma_G)^+ \cup E(\Gamma_G)^-,$$

where $E(\Gamma_G)^+$ and $E(\Gamma_G)^-$ denote respectively the sets of positive and negative edges. Now, we can define a function $\mu : E(\Gamma_G)^+ \rightarrow X^\pm$ as follows: if $e = (v_0, v_1) \in E(\Gamma_G)^+$ then $\mu(e) = \xi(v_1)$. Next, μ can be extended to $E(\Gamma_G)^-$ (and hence to $E(\Gamma_G)$) by setting $\mu(f) = \mu(f^{-1})^{-1}$ for every $f \in E(\Gamma_G)^-$.

Example 6. Let $F = F(X)$ be a free group on X . Hence, F embeds into (coincides with) $CDR(\mathbb{Z}, X)$ and Γ_F with the labeling μ defined above is just a Cayley graph of F with respect to X . That is, Γ_F is a labeled simplicial tree.

The action of G on Γ_G induces the action on $E(\Gamma_G)$ as follows $g \cdot (v_0, v_1) = (g \cdot v_0, g \cdot v_1)$ for each $g \in G$ and $(v_0, v_1) \in E(\Gamma_G)$. It is easy to see that $E(\Gamma_G)^+$ is not closed under the action of G but the labeling is equivariant as the following lemma shows.

Lemma 1. *If $e, f \in E(\Gamma_G)$ belong to one G -orbit then $\mu(e) = \mu(f)$.*

Proof. Let $e = (v_0, v_1) \in E(\Gamma_G)^+$. Hence, there exists $g \in G$ such that $v_0 = \langle \alpha, g \rangle$, $v_1 = \langle \alpha + 1, g \rangle$. Let $f \in G$ and consider the following cases.

Case 1. $c(f^{-1}, g) = 0$

Then $f * g = f \circ g$. If $\alpha = 0$ then $f \cdot v_0 = \langle |f|, f \rangle = \langle |f|, f \circ g \rangle$, and $f \cdot v_1 = \langle |f| + 1, f \circ g \rangle$. Hence, $f \cdot e \in E(\Gamma_G)^+$ and $\mu(f \cdot e) = \xi(f \cdot v_1) = g(1) = \xi(v_1) = \mu(e)$.

Case 2. $c(f^{-1}, g) > 0$

(a) $\alpha + 1 \leq c(f^{-1}, g)$

Then $f \cdot v_0 = \langle |f| + \alpha - 2\alpha, f \rangle = \langle |f| - \alpha, f \rangle$ and $f \cdot v_1 = \langle |f| - (\alpha + 1), f \rangle$. So, $d(\varepsilon, f \cdot v_1) < d(\varepsilon, f \cdot v_0)$ and $f \cdot e \in E(\Gamma_G)^-$. Now,

$$\begin{aligned} \mu(f \cdot e) &= \mu((f \cdot e)^{-1})^{-1} = \mu((f \cdot v_1, f \cdot v_0))^{-1} = \xi(f \cdot v_0)^{-1} = f(|f| - \alpha)^{-1} \\ &= g(\alpha + 1) = \xi(v_1) = \mu(e). \end{aligned}$$

(b) $\alpha = c(f^{-1}, g)$

We have $f \cdot v_0 = \langle |f| - \alpha, f \rangle$ and $f \cdot v_1 = \langle |f| + (\alpha + 1) - 2c(f^{-1}, g), f * g \rangle = \langle |f| - \alpha + 1, f * g \rangle$. It follows that $f \cdot e \in E(\Gamma_G)^+$ and $\mu(f \cdot e) = \xi(f \cdot v_1) = (f * g)(|f| - \alpha + 1)$. At the same time, $f * g = f_1 \circ g_1$, where $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$, so, $(f * g)(|f| - \alpha + 1) = g_1(1) = g(\alpha + 1)$ and $\mu(f \cdot e) = g(\alpha + 1) = \xi(\langle \alpha + 1, g \rangle) = \xi(v_1) = \mu(e)$.

(c) $\alpha > c(f^{-1}, g)$

Hence, $f \cdot v_0 = \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle$ and $f \cdot v_1 = \langle |f| + \alpha + 1 - 2c(f^{-1}, g), f * g \rangle$. Obviously, $f \cdot e \in E(\Gamma_G)^+$ and

$$\begin{aligned} \mu(f \cdot e) &= \xi(f \cdot v_1) = (f * g)(|f| + \alpha + 1 - 2c(f^{-1}, g)) = g_1(\alpha + 1 - c(f^{-1}, g)) \\ &= g(\alpha + 1) = \xi(v_1) = \mu(e), \end{aligned}$$

where $f * g = f_1 \circ g_1$, $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$.

Thus, in all possible cases we got $\mu(f \cdot e) = \mu(e)$ and the required statement follows. \square

Let v, w be two points of Γ_G . Since Γ_G is a Λ -tree there exists a unique geodesic connecting v to w , which can be viewed as a “path” in the following sense. A *path from v to w* is a sequence of edges $p = \{e_\alpha\}$, $\alpha \in [1, d(v, w)]$ such that $o(e_1) = v$, $t(e_{d(v, w)}) = w$ and $t(e_\alpha) = o(e_{\alpha+1})$ for every $\alpha \in [1, d(v, w) - 1]$. In other words, a path is an “edge” counter-part of a geodesic and usually, for the path from v to w (which is unique since Γ_G is a Λ -tree) we are going to use the same notation as for the geodesic between these points, that is, $p = [v, w]$. In the case when $v = w$ the path p is empty. The *length* of p we denote by $|p|$ and set $|p| = d(v, w)$. Now, the *path label* $\mu(p)$ for a path $p = \{e_\alpha\}$ is the function $\mu : \{e_\alpha\} \rightarrow X^\pm$, where $\mu(e_\alpha)$ is the label of the edge e_α .

Lemma 2. *Let v, w be points of Γ_G and p the path from v to w . Then $\mu(p) \in R(\Lambda, X)$.*

Proof. From the definition of Γ_G it follows that the statement is true when $v = \varepsilon$. Let $v_0 = Y(\varepsilon, v, w)$ and let p_v and p_w be the paths from ε respectively to v and w . Also, let p_1 and p_2 be the paths from v_0 to v and w . Since $\mu(p_v), \mu(p_w) \in R(\Lambda, X)$ then $\mu(p_1), \mu(p_2) \in R(\Lambda, X)$ as subwords. Hence, $\mu(p) \notin R(\Lambda, X)$ implies that the first edges e_1 and e_2 correspondingly of p_1 and p_2 have the same label. But this contradicts the definition of Γ_G because in this case $t(e_1) \sim t(e_2)$, but $t(e_1) \neq t(e_2)$. \square

As usual, if p is a path from v to w then its *inverse* denoted p^{-1} is a path from w back to v . In this case, the label of p^{-1} is $\mu(p)^{-1}$, which is again an element of $R(\Lambda, X)$.

Define

$$V_G = \{v \in \Gamma_G \mid \exists g \in G : v = \langle |g|, g \rangle\},$$

which is a subset of points in Γ_G corresponding to the elements of G . Also, for every $v \in \Gamma_G$ let

$$path_G(v) = \{\mu(p) \mid p = [v, w] \text{ where } w \in V_G\}.$$

The following lemma follows immediately.

Lemma 3. *Let $v \in V_G$. Then $path_G(v) = G \subset CDR(\Lambda, X)$.*

The action of G on $E(\Gamma_G)$ extends to the action on all paths in Γ_G , hence, Lemma 1 extends to the case when e and f are two G -equivalent paths in Γ_G .

3 Effective representation by infinite words

In this section we introduce some basic notions concerning effectiveness when dealing with groups of infinite words.

3.1 Infinite words viewed as computable functions

We say that a group $G = \langle Y \rangle$, $Y = \{y_1, \dots, y_m\}$ has an *effective representation* by Λ -words over an alphabet X if $G \subset CDR(\Lambda, X)$ and

- (ER1) each function $y_i : [1, |y_i|] \rightarrow X^\pm$ is computable, that is, one can effectively determine $y_i(\alpha)$ for every $\alpha \in [1, |y_i|]$ and $i \in [1, m]$,
- (ER2) for every $i, j \in [1, m]$ and every $\alpha_i \in [1, |y_i|]$, $\alpha_j \in [1, |y_j|]$ one can effectively compute $c(h_i, h_j)$, where $h_i = y_i^{\pm 1} \upharpoonright_{[\alpha_i, |y_i|]}$, $h_j = y_j^{\pm 1} \upharpoonright_{[\alpha_j, |y_j|]}$.

Observe that since every y_i is computable, y_i^{-1} is computable too for every $i \in [1, m]$. Next, it is obvious that concatenation of computable functions is computable, as well as restriction of a computable function to a computable domain. Thus, if $g_i * g_j = h_i \circ h_j$, where $g_i = y_i^{\delta_i} = h_i \circ c$, $g_j = y_j^{\delta_j} = c^{-1} \circ h_j$, $\delta_i, \delta_j = \pm 1$, then both h_i and h_j are computable as restrictions $h_i = g_i \upharpoonright_{[1, \alpha]}$, $h_j = g_j \upharpoonright_{[\alpha+1, |g_j|]}$ for $\alpha = |c| = c(g_i^{-1}, g_j)$, and so is $g_i * g_j$. Now, using (ER2) twice we can determine $c((g_i * g_j)^{-1}, g_k)$, where $g_k = y_k^{\delta_k}$, $\delta_k = \pm 1$. Indeed, $c((g_i * g_j)^{-1}, g_k) = c(h_j^{-1} \circ h_i^{-1}, g_k)$, so, if $c(h_j^{-1}, g_k) < |h_j^{-1}|$ then $c((g_i * g_j)^{-1}, g_k) = c(h_j^{-1}, g_k)$ which is computable by (ER2), and if $c(h_j^{-1}, g_k) \geq |h_j^{-1}|$ then $c((g_i * g_j)^{-1}, g_k) = |h_j| + c(h_i^{-1}, h_k)$, where $h_k = g_k \upharpoonright_{[|h_j|+1, |g_k|]}$ – again, all components are computable and so is $c((g_i * g_j)^{-1}, g_k)$. It follows that $y_i^{\pm 1} * y_j^{\pm 1} * y_k^{\pm 1}$ is a computable function for every $i, j, k \in [1, m]$. Continuing in the same way by induction one can show that every finite product of elements

from $Y^{\pm 1}$, that is, every element of G given as a finite product of generators and their inverses, is computable as a function defined over a computable segment in Λ to X^{\pm} . Moreover, for any $g, h \in G$ one can effectively find $\text{com}(g, h)$ as a computable function. In particular, we automatically get a solution to the Word Problem in G provided G has effective representation by Λ -words over an alphabet X .

3.2 Effective hierarchy for \mathbb{Z}^n -free groups

Now, consider a finitely generated \mathbb{Z}^n -free group G , where $n \in \mathbb{N}$. Using Bass-Serre theory one can decompose G into a finite graph of groups with \mathbb{Z}^{n-1} -free vertex groups and maximal abelian (in the corresponding vertex groups) edge groups. Continuing this process inductively, one can obtain a finite hierarchy \mathcal{G} of \mathbb{Z}^k -free groups, where $k < n$, such that G can be built from groups in \mathcal{G} by amalgamated free products and HNN-extensions along maximal abelian subgroups (see [31], [2]). At the same time by Theorem 2, G can be embedded into $CDR(\mathbb{Z}^n, X)$ for some alphabet X . Unfortunately, even if we know that the representation of G by \mathbb{Z}^n -words over X is effective it does not give us an effective representation of any $H \in \mathcal{G}$. So, in the case when $\Lambda = \mathbb{Z}^n$ we are going to introduce a stronger version of effective representation which takes into account the hierarchical structure of the group.

Suppose $n > 1$ and consider the \mathbb{Z}^n -tree (Γ_G, d) which arises from the embedding of G into $CDR(\mathbb{Z}^n, X)$.

We say that $p, q \in \Gamma_G$ are \mathbb{Z}^{n-1} -equivalent ($p \sim q$) if $d(p, q) \in \mathbb{Z}^{n-1}$, that is, $d(p, q) = (a_1, \dots, a_n)$, $a_n = 0$. From metric axioms it follows that “ \sim ” is an equivalence relation and every equivalence class defines a \mathbb{Z}^{n-1} -subtree of Γ_G .

Let $\Delta_G = \Gamma_G / \sim$ and denote by ρ the mapping $\Gamma_G \rightarrow \Gamma_G / \sim$. It is easy to see that Δ_G is a simplicial tree. Indeed, define $\tilde{d} : \Delta_G \rightarrow \mathbb{Z}$ as follows:

$$\forall \tilde{p}, \tilde{q} \in \Delta_G : \tilde{d}(\tilde{p}, \tilde{q}) = k \text{ iff } d(p, q) = (a_1, \dots, a_n) \text{ and } a_n = k. \quad (1)$$

From metric properties of d it follows that \tilde{d} is a well-defined metric.

Since G acts on Γ_G by isometries then $p \sim q$ implies $g \cdot p \sim g \cdot q$ for every $g \in G$. Moreover, if $d(p, q) = (a_1, \dots, a_n)$ then $d(g \cdot p, g \cdot q) = (a_1, \dots, a_n)$. Hence, $\tilde{d}(g \cdot \tilde{p}, g \cdot \tilde{q}) = \tilde{d}(\tilde{p}, \tilde{q})$, that is, G acts on Δ_G by isometries, but the action is not free in general. From Bass-Serre theory it follows that $\Psi_G = \Delta_G / G$ is a graph in which vertices and edges correspond to G -orbits of vertices and edges in Δ_G .

Lemma 4. Ψ_G is a finite graph.

Proof. Let $G = \langle g_1, \dots, g_k \rangle$. Let K be a subtree of Γ_G spanned by $g_i^{\pm 1} \cdot \varepsilon$, $i \in [1, k]$. It is easy to see that if $|g_i| = (a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n$, where $a_{in} \geq 0$, $i \in [1, k]$ then $\tilde{K} = \rho(K) \subset \Delta_G$ is a finite subtree such that

$$|V(\tilde{K})| \leq 2 \sum_{i=1}^n a_{in}.$$

Now we claim that for every $q \in \Gamma_G$ there exists $p \in K$ and $g \in G$ such that $q = g \cdot p$. Indeed, since Γ_G is spanned by $g \cdot \varepsilon$, $g \in G$ then let $h \in G$ be such that $p = \langle \alpha, h \rangle$, $h = h_1 \cdots h_m$, where $h_j \in \{g_i^{\pm 1} \mid i \in [1, k]\}$. Then

$$\begin{aligned} [\varepsilon, h \cdot \varepsilon] &\subset [\varepsilon, h_m] \cup [h_{m-1} \cdot \varepsilon, (h_{m-1}h_m) \cdot \varepsilon] \cup \cdots \\ &\cup [(h_1 \cdots h_{m-1}) \cdot \varepsilon, (h_1 \cdots h_m) \cdot \varepsilon]. \end{aligned}$$

It follows that $q \in [(h_j \cdots h_{m-1}) \cdot \varepsilon, (h_j \cdots h_m) \cdot \varepsilon]$ for some j and $(h_j \cdots h_{m-1})^{-1} \cdot q = p \in [\varepsilon, h_m \cdot \varepsilon] \subset K$. So $q = g \cdot p$ for $g = h_j \cdots h_{m-1}$ as required.

From the claim it follows that Γ_G is spanned by translates of K , so Δ_G is spanned by translates of \tilde{K} . Hence, there can be only finitely many G -orbits of vertices and edges in Δ_G , and Ψ_G is a finite graph. \square

From Lemma 4 it follows that the number of G -orbits of \mathbb{Z}^{n-1} -subtrees in Γ_G is finite and equal to $|V(\Psi_G)|$. So, let $|V(\Psi_G)| = m$ and $\mathcal{T}_1, \dots, \mathcal{T}_m$ be these G -orbits.

Consider Ψ_G . The set of vertices and edges of Ψ_G we denote correspondingly by $V(\Psi_G)$ and $E(\Psi_G)$ so that

$$\sigma : E(\Psi_G) \rightarrow V(\Psi_G), \quad \tau : E(\Psi_G) \rightarrow V(\Psi_G), \quad \bar{\cdot} : E(\Psi_G) \rightarrow E(\Psi_G)$$

satisfy the following conditions:

$$\sigma(\bar{e}) = \tau(e), \quad \tau(\bar{e}) = \sigma(e), \quad \bar{\bar{e}} = e, \quad \bar{e} \neq e.$$

Let \mathcal{T} be a maximal subtree of Ψ_G and let $\pi : \Delta_G \rightarrow \Delta_G/G = \Psi_G$ be the canonical projection of Δ_G onto its quotient, so $\pi(v) = Gv$ and $\pi(e) = Ge$ for every $v \in V(\Delta_G)$, $e \in E(\Delta_G)$. There exists an injective morphism of graphs $\eta : \mathcal{T} \rightarrow \Delta_G$ such that $\pi \circ \eta = id_{\mathcal{T}}$ (see Section 8.4 of [8]), in particular $\eta(\mathcal{T})$ is a subtree of Δ_G . One can extend η to a map (which we again denote by η) $\eta : \Psi_G \rightarrow \Delta_G$ such that η maps vertices to vertices, edges to edges, and so that $\pi \circ \eta = id_{\Psi_G}$. Notice, that in general η is not a graph morphism. To this end choose an orientation O of the graph Ψ_G . Let $e \in O - \mathcal{T}$. Then there exists an edge $e' \in \Delta_G$ such that $\pi(e') = e$. Clearly, $\sigma(e')$ and $\eta(\sigma(e))$ are in the same G -orbit. Hence $g \cdot \sigma(e') = \eta(\sigma(e))$ for some $g \in G$. Define $\eta(e) = g \cdot e'$ and $\eta(\bar{e}) = \overline{\eta(e)}$. Notice that vertices $\eta(\tau(e))$ and $\tau(\eta(e))$ are in the same G -orbit. Hence there exists an element $\gamma_e \in G$ such that $\gamma_e \cdot \tau(\eta(e)) = \eta(\tau(e))$.

Put

$$G_v = \text{Stab}_G(\eta(v)), \quad G_e = \text{Stab}_G(\eta(e))$$

and define boundary monomorphisms as inclusion maps $i_e : G_e \hookrightarrow G_{\sigma(e)}$ for edges $e \in \mathcal{T} \cup O$ and as conjugations by $\gamma_{\bar{e}}$ for edges $e \notin \mathcal{T} \cup O$, that is,

$$i_e(g) = \begin{cases} g, & \text{if } e \in \mathcal{T} \cup O, \\ \gamma_{\bar{e}} g \gamma_{\bar{e}}^{-1}, & \text{if } e \notin \mathcal{T} \cup O. \end{cases}$$

According to the Bass-Serre structure theorem we have

$$G \simeq \pi(\mathcal{G}, \Psi_G, \mathcal{T}) = \langle G_v \ (v \in V(\Psi_G)), \ \gamma_e \ (e \in E(\Psi_G)) \mid \text{rel}(G_v), \quad (2)$$

$$\gamma_e i_e(g) \gamma_e^{-1} = i_{\bar{e}}(g) \ (g \in G_e), \ \gamma_e \gamma_{\bar{e}} = 1, \ \gamma_e = 1 \ (e \in \mathcal{T}) \rangle.$$

Let $\mathcal{K} = \rho^{-1}(\eta(\mathcal{T}))$, $\overline{\mathcal{K}} = \rho^{-1}(\eta(\Psi_G))$, hence, \mathcal{K} , $\overline{\mathcal{K}}$ are subtrees of Γ_G such that $\mathcal{K} \subseteq \overline{\mathcal{K}}$. Obviously $T_0 \subseteq \mathcal{K}$. Moreover, both \mathcal{K} and $\overline{\mathcal{K}}$ contain finitely many \mathbb{Z}^{n-1} -subtrees, and meet every G -orbit of \mathbb{Z}^{n-1} -subtrees of Γ_G .

For every $v \in \Psi_G$ we have $\text{Stab}_G(\eta(v)) = \text{Stab}_G(T_{\eta(v)})$, where $\eta(v) = \rho(T_{\eta(v)})$. Denote by T_0 the \mathbb{Z}^{n-1} -subtree containing ε . Obviously, $\text{Stab}_G(T_0)$ is a subgroup of $\text{CDR}(\mathbb{Z}^{n-1}, X)$.

Lemma 5. *Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} . Then*

$$\text{Stab}_G(T) = f_T * K_T * f_T^{-1},$$

where K_T is a subgroup of $\text{CDR}(\mathbb{Z}^{n-1}, X)$ (possibly trivial) and $f_T = \mu([\varepsilon, x_T]) \in \text{CDR}(\mathbb{Z}^n, X)$. Moreover, is $\text{Stab}_G(T)$ is not trivial then $x_T \in \text{Axis}(g) \cap T$ for some $g \in \text{Stab}_G(T)$.

Proof. If $\text{Stab}_G(T)$ is trivial then the statement obviously holds.

Suppose $\text{Stab}_G(T) \neq 1$ and let $g \in \text{Stab}_G(T)$. By Corollary 1.6 [6], $\text{Axis}(g)$ meets every $\langle g \rangle$ -invariant subtree of Γ_G . Since T is $\langle g \rangle$ -invariant then $\text{Axis}(g) \cap T \neq \emptyset$. Hence, choose some $x_T \in \text{Axis}(g) \cap T$ and put $f_T = \mu([\varepsilon, x_T])$. We have $g \cdot x_T \in T$, so $|f_T^{-1} * g * f_T| \in \mathbb{Z}^{n-1}$, in other words, $g = f_T * a_g * f_T^{-1}$, $a_g \in \text{CDR}(\mathbb{Z}^{n-1}, X)$. Since $\text{Stab}_G(T)$ is a group then $K_T = \{a_g \mid g \in \text{Stab}_G(T)\}$ is a subgroup of $\text{CDR}(\mathbb{Z}^{n-1}, X)$. □

Let e be an edge of Ψ_G such that $e \in O$, $e \notin \mathcal{T}$. Let $v = \sigma(\eta(e)) = \eta(\sigma(e))$, $w = \tau(\eta(e))$ and $u = \eta(\tau(e)) = \gamma_e \cdot w$. We have $u, v \in \eta(\mathcal{T})$, $w \notin \eta(\mathcal{T})$. Hence,

$$\gamma_e \text{Stab}_G(w) \gamma_e^{-1} = \text{Stab}_G(u).$$

By definition we have $i_e(G_e) \subseteq G_v = \text{Stab}_G(T)$, where $T = \rho^{-1}(v)$ and $i_{\bar{e}}(G_{\bar{e}}) = \gamma_e G_e \gamma_e^{-1} \subseteq G_u = \text{Stab}_G(S)$, where $S = \rho^{-1}(u)$. Thus, we have $i_e(G_e) = f_T * A * f_T^{-1}$, $i_{\bar{e}}(G_{\bar{e}}) = f_S * B * f_S^{-1}$, where $A \leq K_T$ and $B \leq K_S$ are isomorphic abelian subgroups of $\text{CDR}(\mathbb{Z}^{n-1}, X)$. So,

$$\gamma_e * (f_T * A * f_T^{-1}) * \gamma_e^{-1} = f_S * B * f_S^{-1}$$

and it follows that $f_S^{-1} * \gamma_e * f_T = r_e \in \text{CDR}(\mathbb{Z}^n, X)$ so that $r_e * A * r_e^{-1} = B$. Thus, we have

$$\gamma_e = f_S * r_e * f_T^{-1}.$$

Observe that $r_e \in \text{CDR}(\mathbb{Z}^n, X) - \text{CDR}(\mathbb{Z}^{n-1}, X)$ because otherwise $\gamma_e \cdot T = S$, that is, $u = v$, $S = T$ and thus $\gamma_e \in \text{Stab}_G(T)$ - a contradiction.

Now, we can give an inductive definition of effective hierarchy for a finitely generated \mathbb{Z}^n -free group G . We say that G has an *effective hierarchy over an alphabet X* if the following conditions are satisfied.

- (EFH1) If $n = 1$ then G has effective representation by \mathbb{Z}^n -words over the alphabet X .
- (EFH2) If $n > 1$ then in the presentation (2) for G
- (a) each vertex group G_v is given in the form $f_v * K_v * f_v^{-1}$, where K_v has effective hierarchy over X (we assume that effective hierarchy is defined for K_v by induction) and f_v is a computable \mathbb{Z}^n -word over X ,
 - (b) for each edge group G_e , its images in the corresponding vertex groups have effective presentations over X ,
 - (c) each r_e in the representation $\gamma_e = f_S * r_e * f_T^{-1}$ is given as a computable \mathbb{Z}^n -word over X .

Observe that from the definition above it follows that effective hierarchy over X implies effective representation over X .

4 Effective regular completions

Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$, where \mathbb{Z}^n is ordered with respect to the right lexicographic order. Here we do not assume X to be finite. We are going to construct a finite alphabet Y and a finitely generated group H which is subgroup of $CDR(\mathbb{Z}^n, Y)$ such that the length function on H induced from $CDR(\mathbb{Z}^n, Y)$ is regular and G embeds into H so that the length is preserved by the embedding. In other words, we are going to construct a finitely generated \mathbb{Z}^n -completion of G (see [19]). Moreover, if G has an effective hierarchy over X then we show that the construction of H is effective and it has an effective hierarchy over Y .

The argument is conducted by induction on n .

4.1 Simplicial case

Let G be a finitely generated subgroup of $CDR(\mathbb{Z}, X)$. Hence, Γ_G is a simplicial tree and $\Delta = \Gamma_G/G$ is a folded X -labeled digraph (see [14]) with labeling induced from Γ_G . Δ is finite which follows from the fact that G is finitely generated and from the construction of Γ_G . Moreover, Δ recognizes G with respect to some vertex v (the image of ε) in the sense that $g \in CDR(\mathbb{Z}, X)$ belongs to G if and only if there exists a loop in Δ at v such that its label is exactly g .

The following lemma provides the required result.

Lemma 6. *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}, X)$. Then there exists a finite alphabet Y and an embedding $\phi : G \rightarrow H$, where $H = F(Y)$, inducing an embedding $\psi : \Gamma_G \rightarrow \Gamma_H$ such that*

- (i) $|g|_G = |\phi(g)|_H$ for every $g \in G$,

- (ii) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H ,
- (iii) if a and b are non- G -equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non- H -equivalent ends of Γ_H ,
- (iv) if A and B are maximal abelian subgroups of G which are not conjugate in G then $\phi(A)$ and $\phi(B)$ are not conjugate in H .

Moreover, if G has an effective representation by \mathbb{Z} -words over X then Y can be found effectively and the embedding $\phi : G \rightarrow H$ is effective.

Proof. Since G is finitely generated then there are only finitely many letters which are used in the representation of the generating set of G , so, X may be considered to be finite. Consider $\Delta = \Gamma_G/G$ and let $v \in V(\Delta)$ be the image of ε . Let $E = \{e_1, \dots, e_k\}$ be the set of edges of Δ and E_+ an orientation of Δ . Take a copy of Δ denoted Δ' , which has the set of edges E' and orientation E'_+ corresponding to E and E_+ . Let $v' \in V(\Delta')$ correspond to $v \in V(\Delta)$. Introduce a labeling function μ' on edges of Δ' as follows: $\mu'(e_i) = e_i$ if $e_i \in E'_+$, and $\mu'(e_i) = e_i^{-1}$ if $e_i \in E' - E'_+$. Hence, $\mu' : E' \rightarrow E'$ and as a result we get a E' -labeled digraph Δ' . There exists a natural isomorphism of graphs $\gamma : \Delta \rightarrow \Delta'$ which induces a natural isomorphism $\phi : G \rightarrow G'$, where $G' \leq F(E')$ is recognized by Δ' with respect to $v' \in V(\Delta')$. Let $Y = E'$ and $H = F(Y)$. Since G' is a subgroup of $F(Y)$ then $\Gamma_{G'}$ naturally embeds into Γ_H which is a Cayley graph of H . Now, $\phi : G \rightarrow G'$ induces the isomorphism between Γ_G and $\Gamma_{G'}$ which gives an embedding $\psi : \Gamma_G \rightarrow \Gamma_H$.

Observe that under assumption that G has an effective representation by \mathbb{Z} -words over X , one can effectively enumerate X and Δ can be constructed effectively from a finite wedge of loops labeled by finite words (which again can be found effectively) corresponding to the generators of G . Hence, G' has an effective representation by \mathbb{Z} -words over a finite alphabet Y (found effectively) meaning that the embedding $G' \hookrightarrow H$ is effective. Hence, the embedding $G \hookrightarrow H$ is effective too. Now, we prove the required properties of this embedding.

First of all, obviously $|g|_G = |\phi(g)|_{G'}$ for every $g \in G$. Hence, $|g|_G = |\phi(g)|_H$ for every $g \in G$ and (i) follows.

Next, if $g \in G$ is not a proper power in G then $\phi(g)$ is not a proper power in $F(Y)$. Indeed, if $\phi(g) \in G'$ is a proper power in $F(Y)$ then, due to one-to-one correspondence between edges of Δ' and their labels, there exists a path $q = q_1 q_2 q_1^{-1} \in \Delta'$ at v' such that $\mu'(q_2)$ is cyclically reduced, $q_2 = q_3^m$, $m > 1$ for some loop q_3 at $t(q_1)$, and $\mu'(p) = \theta(g)$. Hence, $\phi(g)$ is a proper power in G' and $g = \mu(\gamma^{-1}(p))$ is a proper power in Δ . So, (ii) is proved.

Let us prove (iii). Every end a of Γ_G corresponds to a unique infinite geodesic ray r_a in Γ_G originating at ε whose edges are labeled by X^\pm , and r_a corresponds to a reduced infinite path p_a in Δ starting at v . Ends a and b of Γ_G are $F(X)$ -equivalent if and only if $\mu(p_b) = \mu(p) \circ \mu(p_a)$, where p is a reduced path. Correspondingly, a and b are G -equivalent if and only if $\mu(p_b) = \mu(p) \circ \mu(p_a)$, where p is a reduced loop at v , that is, if $\mu(p) \in G$. Since Δ is folded then it follows that for G -equivalent ends a and b we have $p_b = pp_a$.

Now, let $\psi(a)$ and $\psi(b)$ be two ends of $\psi(\Gamma_G)$. Hence, they correspond to unique reduced infinite paths $\psi(p_a)$ and $\psi(p_b)$ in Δ' starting at v' . If $\psi(a)$ and $\psi(b)$ are H -equivalent then $\mu'(p_{\psi(b)}) = \mu'(\psi(p)) \circ \mu(p_{\psi(a)})$, where $\psi(p)$ is a reduced path in Δ' . But since there exists one-to-one correspondence between edges of Δ' and their labels then it follows that $p_{\psi(b)} = \psi(p)p_{\psi(a)}$ and $\psi(p)$ is a loop at v' . That is, $\psi(a)$ and $\psi(b)$ are G' -equivalent which implies that a and b are G -equivalent. So, (iii) follows.

Finally, to prove (iv) observe that a maximal abelian subgroup C of G corresponds to a pair of ends of Γ_G which are the ends of its axis $Axis(C)$. Now, (iv) follows from (iii). \square

Lemma 6 can be generalized to the following result.

Corollary 1. *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}, X)$. Assume that Γ_G is embedded into a \mathbb{Z} -tree T whose edges are labeled by X^\pm so that the action of G on Γ_G extends to an action of G on T , and there are only finitely many G -orbits of ends of T which belong to $T - \Gamma_G$. Then there exists a finite alphabet Y , a \mathbb{Z} -tree T' whose edges are labeled by Y^\pm , and a finitely generated subgroup $H \subseteq CDR(\mathbb{Z}, Y)$ such that Γ_H is embedded into T' so that the action of H on Γ_H extends to a regular action of H on T' . Also, there is an embedding $\theta : T \rightarrow T'$, where $\theta(\Gamma_G) \subseteq \Gamma_H$, which induces an embedding $\phi : G \rightarrow H$ such that*

- (i) $|g|_G = |\phi(g)|_H$ for every $g \in G$,
- (ii) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H ,
- (iii) if a and b are non- G -equivalent ends of T then $\theta(a)$ and $\theta(b)$ are non- H -equivalent ends of T' .

Moreover, if

- (e1) G has an effective representation by \mathbb{Z} -words over X ,
- (e2) a set of representatives q_1, \dots, q_m of G -orbits of ends of T which belong to $T - \Gamma_G$ is given as a set of functions $q_i : [1, \infty) \rightarrow X^\pm$ so that each q_i is computable,

then Y can be found effectively, H has an effective representation by \mathbb{Z} -words over Y , and the embedding $\phi : G \rightarrow H$ is effective.

Proof. Consider T/G . Since there are only finitely many G -orbits of ends in $T - \Gamma_G$ then T/G consists of $\Delta = \Gamma_G/G$ and a forest formed by finitely many infinite rays attached to some vertices of Δ . These rays correspond to G -orbits of ends in $T - \Gamma_G$. By Lemma 6 there exists a finite alphabet Y_1 and a relabeling of edges of Δ by Y_1^\pm which induces embeddings $\psi : \Gamma_G \rightarrow \Gamma_{F(Y_1)}$ and $\phi : G \rightarrow F(Y_1)$ in such a way that

- (a) $|g|_G = |\phi(g)|_{F(Y_1)}$ for every $g \in G$,
- (b) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of $F(Y_1)$,
- (c) if a and b are non- G -equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non- $F(Y)$ -equivalent ends of $\Gamma_{F(Y_1)}$.

Now, the new labeling of Δ can be extended to a labeling of T/G as follows. Since there are only finitely many infinite rays in T/G then there are only finitely many branch-points in $T/G - \Delta$. Hence, in $T/G - \Delta$ there are finitely many paths $p_{v,w}$ of the form $[v, w]$, where v is either a branch-point or a vertex of Δ which an infinite ray from $T/G - \Delta$ is attached to, and w is an adjacent to v branch-point. Denote this set by P . Also, there are finitely many infinite rays $R = \{r_1, \dots, r_m\}$ in $T/G - \Delta$ which do not contain any branch-points. Eventually, every infinite ray r in T/G corresponding to a G -orbit of ends in $T - \Gamma_G$, attached to a vertex u_a in Δ can be decomposed as

$$p_{u_a, v_1}, \dots, p_{v_{k-1}, v_k}, r_i$$

where $p_{u_a, v_1}, \dots, p_{v_{k-1}, v_k} \in P$ and $r_i \in R$. Now for each $p \in P \cup R$ relabel all edges of p by the same letter a_p so that $a_p \neq a_q$ whenever $p \neq q$, $p, q \in P \cup R$, and let $Y_2 = \{a_p \mid p \in P \cup R\}$. Observe that Y_2 is finite even if X is not finite. Let $W = \{\mu(p) \mid p \in P\}$. Hence, the label of any infinite ray in $T/G - \Delta$ can be decomposed as

$$w_1 \circ w_2 \circ \dots \circ w_k \circ \mu(r)$$

for some $w_i \in P$, $r \in R$.

Let $Y = Y_1 \cup Y_2$ and $H = \langle Y_1 \cup W \rangle \subset CDR(\mathbb{Z}, Y)$. Let Q be a collection of paths of the form pr , where $\mu(p) \in H$ and either $r \in R$ or r is empty. Define $T' = Q / \sim$, where “ \sim ” stands for identification of common initial subwords for every pair $p_1, p_2 \in Q$. Hence, T' is a \mathbb{Z} -tree labeled by Y containing Γ_H , on which H acts regularly by left multiplication. T' contains a copy of T , relabeled as shown above, which provides an embedding $\theta : T \rightarrow T'$, where $\theta(\Gamma_G) \subseteq \Gamma_H$. Finally, non- G -equivalent ends of T are labeled by different letters in $\theta(T)$, so combined with (c) it implies (iii).

Assuming (e1) it follows that Y_1 can be found effectively by Lemma 6. Next, using (e2) we can find all branch-points in $T/G - \Delta$ effectively. Indeed, each q_i can be effectively represented in the form $g_i \circ q'_i$, where $g_i \in G$ and q'_i is an infinite ray in $T/G - \Delta$ (g_i can be found by “reading” the label of q_i in Δ letter by letter, the process stops because g_i exists). Thus one can find out effectively if q'_i and q'_j originate from the same vertex in Δ and, in the case they do, determine their maximal common initial segment which gives a branch-point (since q_i and q_j are not G -equivalent for $i \neq j$ then the maximal common initial segment of q'_i and q'_j is finite). The number of branch-points is bounded by the number of orbits of infinite rays in T/G , so eventually one can find all of them. Hence, Y_2 and W can be found effectively, so H has an effective representation

by \mathbb{Z} -words over Y and effectiveness of the embedding of G into H follows from the effectiveness of $\phi : G \rightarrow F(Y_1)$. \square

4.2 General case

Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ for some alphabet X . We are going to use the notations introduced in Subsection 3.2, that is, we assume that \mathcal{K} , Ψ_G , Δ_G etc. are defined for G as well as the presentation (2).

First of all, we relabel Γ_G so that non- G -equivalent \mathbb{Z}^{n-1} -subtrees are labeled by disjoint alphabets.

Recall that every edge e in Γ_G is labeled by a letter $\mu(e) \in X^\pm$. Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} and X_T a copy of X (disjoint from X) so that we have a bijection $\pi_T : X \rightarrow X_T$, where $\pi_T(x^{-1}) = \pi_T(x)^{-1}$ for every $x \in X$. We assume $X_S \cap X_T = \emptyset$ for distinct $S, T \in \mathcal{K}$. Let Γ' be a copy of Γ_G and $\nu : \Gamma' \rightarrow \Gamma_G$ a natural bijection (the bijection on points naturally induces the bijection on edges). Denote $\varepsilon' = \nu^{-1}(\varepsilon)$.

Let $X' = \bigcup \{X_T \mid T \in \mathcal{K}\}$. We introduce a labeling function $\mu' : E(\Gamma') \rightarrow X'^\pm$ as follows: $\mu'(e) = \pi_T(\mu(\nu(e)))$ if $\nu(e) \in T$. μ' naturally extends to the labeling of paths in Γ' . Now, if $V' = \nu^{-1}(V_G)$ then define

$$G' = \{\mu'(p) \mid p = [\varepsilon', v'] \text{ for some } v' \in V'\}.$$

Lemma 7. *G' is a subgroup of $CDR(\mathbb{Z}^n, X')$ which acts freely on Γ' and there exists an isomorphism $\phi : G \rightarrow G'$ such that $L_\varepsilon(g) = L_{\varepsilon'}(\phi(g))$. Moreover, if G has an effective hierarchy over X then G' has an effective hierarchy over X' .*

Proof. Take $g \in G$. Since $g = \mu([\varepsilon, v])$ for some $v \in \Gamma_G$ then define

$$\phi(g) = \mu'([\varepsilon', v']) \in G',$$

where $v' = \nu^{-1}(v)$. All the required properties of G' follow immediately.

The effectiveness part is obvious. \square

According to Lemma 7 we have $\Gamma' = \Gamma_{G'}$. Observe that the structure of \mathbb{Z}^{n-1} -trees in $\Gamma_{G'}$ is the same as in Γ_G . Hence, if “ \sim ” is a \mathbb{Z}^{n-1} -equivalence of points of $\Gamma_{G'}$ then $\Delta_{G'} = \Gamma_{G'}/\sim$ and $\Psi_{G'} = \Delta_{G'}/G'$ are naturally isomorphic respectively to $\Delta_G = \Gamma_G/\sim$ and $\Psi_G = \Delta_G/G$. So, with a slight abuse of notation let $X = X'$, $G = G'$.

Next, we would like to refine the labeling so as to make the alphabet X finite. To do this we have to analyze the structure of the \mathbb{Z}^{n-1} -subtrees of \mathcal{K} .

Lemma 8. *Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} such that $\text{Stab}_G(T)$ is trivial. Then T contains only finitely many branch-points and each branch-point of T is of the form $Y(\varepsilon, x, y)$, where $x, y \in \{x_S \mid S \in \mathcal{K}\}$, $\gamma_e^{\pm 1} \cdot \varepsilon$ ($e \in \Psi_G$). Moreover, if G has an effective hierarchy over X then all branch-points of T can be found effectively.*

Proof. Suppose a is a branch-point of T . Then there exist $x_1, x_2 \in \Gamma_G - T$ such that $a = Y(\varepsilon, x_1, x_2)$. Indeed, otherwise, from the construction of Γ_G there exist distinct $g, h \in G$ such that $g \cdot \varepsilon, h \cdot \varepsilon \in T$, so $g^{-1} * h \in \text{Stab}_G(T)$ - a contradiction. Without loss of generality we can assume that $x_1 \in S_1, x_2 \in S_2$, where S_1 and S_2 are \mathbb{Z}^{n-1} -subtrees of Γ_G adjacent to T . Observe that S_1 and S_2 belong to distinct G -orbits because $\text{Stab}_G(T)$ is trivial. Thus, the number of branch-points in T is finite. Finally, the pair (T, S_i) , $i = 1, 2$ corresponds to an edge $e_i = (\pi(\rho(T)), \pi(\rho(S_i)))$ of Ψ_G . If $e_i \in \mathcal{T}$ then x_i can be chosen to be x_{S_i} , otherwise x_i can be chosen to be $\gamma_e \cdot \varepsilon$.

The effectiveness part of the statement follows immediately. \square

In particular, from Lemma 8 it follows that every \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with trivial stabilizer can be relabeled by a finite alphabet. Indeed, T may be cut at its branch-points into finitely many closed segments and half-open rays which do not contain any branch-points. Then all these segments and rays can be labeled by different letters (all points in each piece is labeled by one letter).

In the case of non-trivial stabilizer the situation is a little more complicated.

Lemma 9. *Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} such that $\text{Stab}_G(T) = f_T * K_T * f_T^{-1}$ is non-trivial. Then Γ_{K_T} embeds into T (the base-point of Γ_{K_T} is identified with x_T), the action of K_T on Γ_{K_T} extends to the action of K_T on T and the following hold*

- (a) *every end of T which does not belong to Γ_{K_T} is K_T -equivalent to one of the ends of a finite subtree which is the intersection of T and the segments $[\varepsilon, x_S]$, $S \in \mathcal{K}$,*
- (b) *every end a of T which does not belong to Γ_{K_T} extends the axis of some centralizer C_a of K_T ,*
- (c) *there are only finitely many K_T -orbits of branch-points of T which do not belong to Γ_{K_T} ,*
- (d) *if $K_T \subset \text{CDR}(\mathbb{Z}^{n-1}, Y)$ for some finite alphabet Y then the labeling of Γ_{K_T} by Y can be K_T -equivariantly extended to a labeling of T by a finite extension Y' of Y .*

Moreover, if G has an effective hierarchy over X then in (b) the centralizer C_a can be found effectively, in (c) representatives of K_T -orbits of branch-points of T which do not belong to Γ_{K_T} can be found effectively, and in (d) the new alphabet Y' can be found effectively provided Y is given.

Proof. (a) follows immediately from the structure of Γ_G explained in detail in Subsection 3.2.

(b), (c) It follows from (a) that there exist finitely many K_T -orbits of ends of T which do not belong to Γ_{K_T} . Fix some representatives a_1, \dots, a_k of these orbits. Let $a \in \{a_1, \dots, a_k\}$. Observe that for each $g \in K_T$, the intersection

$[x_T, a) \cap [x_T, g \cdot a)$ is either a closed segment in T or it is equal to $[x_T, a)$. Consider the former case. Let y be the terminal point of the intersection and let $w_a = \mu([x_T, y])$. If $y \in \Gamma_{K_T}$ then $[x_T, y] \subset \Gamma_{K_T}$. Suppose $y \notin \Gamma_{K_T}$. Hence, $w_a^{-1} * (g * w_a)$ is defined in $CDR(\mathbb{Z}^n, X)$ and either w_a^{-1} or $g * w_a$ cancels completely in the product. By the assumption, y is a branch-point of T which does not belong to Γ_{K_T} . Hence, $ht(g) < ht(w_a)$ and by Lemma 8 and Lemma 9 [19] it follows that $ht(w_a) > ht(g_1)$ for every $g_1 \in C_{K_T}(g)$ and w_a conjugates $C_{K_T}(g)$ to another centralizer in $CDR(\mathbb{Z}^{n-1}, X)$. Moreover, if for some $h \in K_T$ the intersection $[x_T, a) \cap [x_T, f \cdot a)$ is a closed segment in T then $h \in C_{K_T}(g)$ and the end-point y_h of this intersection belongs to the orbit of y under the action of K_T (in fact, under the action of $C_{K_T}(g)$).

Hence, every representative a from the list a_1, \dots, a_k can be associated with a centralizer C_a of K_T (C_a may be trivial) and a branch point x_a on $[x_T, a)$ so that if $x = Y(x_T, a, g \cdot a)$ and $x \in T - \Gamma_{K_T}$ then $x \in G \cdot x_a$.

Next, consider $a, b \in \{a_1, \dots, a_k\}$. Suppose there exist $g, h \in K_T$ such that $[x_T, g \cdot a) \cap [x_T, h \cdot b)$ is a closed segment in T and the end-point x of this segment (other than x_T) does not belong to Γ_{K_T} . Now if $w = \mu([x_T, x])$ then

$$ht(g^{-1}C_a g), ht(h^{-1}C_b h) < ht(w)$$

and from Lemma 9 [19] it follows that $g^{-1}C_a g = h^{-1}C_b h$. Now, if there exists $f \in K_T$ such that $[x_T, g \cdot a) \cap [x_T, f \cdot b)$ is a closed segment in T and the end-point y of this segment (other than x_T) does not belong to Γ_{K_T} then $y \in G \cdot x$.

Finally, (c) follows from (b). Indeed, for each representative a in the list $\{a_1, \dots, a_k\}$ let Y_a be a copy of the alphabet Y so that $Y_a \cap Y_b = \emptyset$ if $a \neq b$. Then $[x_T, a)$ contains an open subinterval which is the axis of a centralizer D_a of $CDR(\mathbb{Z}^{n-1}, X)$ and w_a conjugates $Axis(C_a)$ to $Axis(D_a)$. Thus $Axis(D_a)$ can be relabeled by Y_a and this labeling can be extended to the whole interval $[x_T, a) - \Gamma_{K_T}$, where $([x_T, a) - \Gamma_{K_T}) - Axis(D_a)$ can be labeled arbitrarily by Y_a since it does not contain branch-points. Eventually, one can K_T -equivariantly extend the labeling from each $[x_T, a) - \Gamma_{K_T}$ to its images under the action of K_T .

The effectiveness part easily follows. Indeed, the centralizers of K_T associated to the generators $\gamma_e^{\pm 1}, e \in \Psi_G$ are a part of the effective hierarchy for G . Every end a of T which does not belong to Γ_{K_T} is K_T -conjugate to b which is defined by some $\gamma_e^{\pm 1}$. Hence, if $a = g \cdot b$ then $C_a = gC_b g^{-1}$ and C_b is a part of the effective hierarchy. Next, the intersection of T with a finite subtree of Γ_G formed by the paths corresponding to $\gamma_e^{\pm 1}, e \in \Psi_G$ can be found effectively. Hence, all branch-points in this intersection (which is now a subtree of T) can also be found effectively and every branch-point of T which does not belong to Γ_{K_T} is K_T conjugate to one of these. Finally, in (d), if Y is given effectively then from the construction above it follows that Y' is a disjoint union of several copies of Y , so it can be found effectively as we.

□

Corollary 2. *If G is a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ then X can be taken to be finite.*

Proof. Follows from Lemma 8 and Lemma 9. \square

For a non-linear \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with a non-trivial stabilizer let $\mathcal{B}(\mathcal{T})$ be the set of representatives of branch-points of $T - \Gamma_{K_T}$. By Lemma 9, $\mathcal{B}(\mathcal{T})$ is finite and every branch-point of T which does not belong to Γ_{K_T} is K_T -equivalent to a branch-point from $\mathcal{B}(\mathcal{T})$. Let

$$\mathcal{D}(T) = \{\mu([x_T, y]) \mid y \in \mathcal{B}(\mathcal{T})\}.$$

Observe that $\mathcal{D}(T)$ is a finite subset of $CDR(\mathbb{Z}^{n-1}, X)$.

Let $g \in G$. Hence, $[\varepsilon, g \cdot \varepsilon]$ meets finitely many \mathbb{Z}^{n-1} -subtrees T_0, T_1, \dots, T_k , where $T(g)_0 = T_0$ and T_i is adjacent to T_{i-1} for each $i \in [1, k]$. Observe that T_0 is \mathbb{Z}^{n-1} -subtree of \mathcal{K} . We have

$$[\varepsilon, g \cdot \varepsilon] \subseteq [x_{T_0}, x_{T_1}] \cup \dots \cup [x_{T_{k-1}}, x_{T_k}] \cup [x_{T_k}, g \cdot \varepsilon].$$

Now, there exists $g_0 \in \text{Stab}_G(T_0)$ and a \mathbb{Z}^{n-1} -subtree S_1 of \mathcal{K} adjacent to T_0 such that $T_1 = g_0 \cdot S_1$. Next, there exists $g_1 \in \text{Stab}_G(T_1)$ and a \mathbb{Z}^{n-1} -subtree S_2 of \mathcal{K} adjacent to S_1 such that $T_2 = (g_1 g_0) \cdot S_2$, and so on. After k steps we find a sequence of \mathbb{Z}^{n-1} -subtrees S_0, S_1, \dots, S_k from \mathcal{K} , where $S_0 = T_0$, S_i is adjacent to S_{i-1} , $i \in [1, k]$ and $T_i = (g_{i-1} \dots g_0) \cdot S_i$, where $g_i \in \text{Stab}_G(T_i)$. Hence,

$$\begin{aligned} [\varepsilon, g \cdot \varepsilon] &\subseteq [x_{T_0}, g_0 \cdot x_{T_0}] \cup [g_0 \cdot x_{T_0}, g_0 \cdot x_{S_1}] \cup [g_0 \cdot x_{S_1}, x_{T_1}] \cup [x_{T_1}, (g_1 g_0) \cdot x_{S_1}] \\ &\cup [(g_1 g_0) \cdot x_{S_1}, (g_1 g_0) \cdot x_{S_2}] \cup \dots \cup [(g_{k-1} \dots g_0) \cdot x_{S_{k-1}}, (g_{k-1} \dots g_0) \cdot x_{S_k}] \\ &\cup [(g_{k-1} \dots g_0) \cdot x_{S_k}, x_{T_k}] \cup [x_{T_k}, (g_k \dots g_0) \cdot x_{S_k}], \end{aligned}$$

where $(g_k \dots g_0) \cdot x_{S_k} = g \cdot \varepsilon$.

Since

$$\mu([p, q]) = \mu(g \cdot [p, q]) = \mu([g \cdot p, g \cdot q])$$

and

$$[(g_{i-1} \dots g_0) \cdot x_{S_{i-1}}, (g_{i-1} \dots g_0) \cdot x_{S_i}] = (g_{i-1} \dots g_0) \cdot [x_{S_{i-1}}, x_{S_i}]$$

for $i \in [1, k]$, then

$$\mu([(g_{i-1} \dots g_0) \cdot x_{S_{i-1}}, (g_{i-1} \dots g_0) \cdot x_{S_i}]) = \mu([x_{S_{i-1}}, x_{S_i}]).$$

Also, observe that for any $i \in [1, k]$

$$[(g_{i-1} \dots g_0) \cdot x_{S_i}, x_{T_i}] \cup [x_{T_i}, (g_i \dots g_0) \cdot x_{S_i}]$$

is a path in T_i , where $(g_{i-1} \dots g_0) \cdot x_{S_i}$ and $(g_i \dots g_0) \cdot x_{S_i}$ are $\text{Stab}_G(T_i)$ -equivalent to x_{T_i} . So, it follows that

$$\mu([x_{T_i}, (g_{i-1} \dots g_0) \cdot x_{S_i}]) = f_i \in K_{T_i}, \quad \mu([x_{T_i}, (g_i \dots g_0) \cdot x_{S_i}]) = h_i \in K_{T_i}.$$

Also, observe that $g_0 = \mu([x_{T_0}, g_0 \cdot x_{T_0}])$. Eventually, we have

$$g = g_0 * c_{S_0, S_1} * (f_1^{-1} * h_1) * c_{S_1, S_2} * \cdots * c_{S_{k-1}, S_k} * (f_k^{-1} * h_k),$$

where c_{S_{i-1}, S_i} is the label of the path $[x_{S_{i-1}}, x_{S_i}]$ and the product on the right-hand side is defined in $CDR(\mathbb{Z}^n, X)$.

Now we are ready to perform the inductive step.

Theorem 3. *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ (assume that \mathcal{K} , Ψ_G , Δ_G etc. are defined for G as above). Suppose that for every non-linear \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with a non-trivial stabilizer there exists*

- (a) *an alphabet $Y(T)$,*
- (b) *a \mathbb{Z}^{n-1} -tree T' whose edges are labeled by $Y(T)$,*
- (c) *a finitely generated group $H_T \subset CDR(\mathbb{Z}^{n-1}, Y(T))$*

such that Γ_{H_T} is embedded into T' and the action of H_T on Γ_{H_T} extends to a regular action of H_T on T' . Moreover, assume that there is an embedding $\psi_T : T \rightarrow T'$, where $\psi_T(\Gamma_{K_T}) \subseteq \Gamma_{H_T}$, which induces an embedding $\phi_T : K_T \rightarrow H_T$, and such that if a and b are non- K_T -equivalent ends of T then $\psi_T(a)$ and $\psi_T(b)$ are non- H_T -equivalent ends of $\psi_T(T)$.

Then there exists an embedding of $\mathcal{D}(T)$, $T \in \mathcal{K}$ into $CDR(\mathbb{Z}^n, Y)$, where Y is a finite alphabet containing $\bigcup_{T \in \mathcal{K}} Y(T)$ such that

- (i) *$\{H(T), \mathcal{D}(T), \{c_{x_T, x_S} \mid S \text{ is adjacent to } T \text{ in } \mathcal{K}\} \mid T \in \mathcal{K}\}$ generates a group H of $CDR(\mathbb{Z}^n, Y)$ which acts regularly on Γ_H with respect to ε_H ,*
- (ii) *there exists an embedding $\psi : \Gamma_G \rightarrow \Gamma_H$, $\psi(\varepsilon_G) = \varepsilon_H$ which induces an embedding $\phi : G \rightarrow H$, such that if a and b are non- G -equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non- H -equivalent ends of $\psi(\Gamma_G)$.*

Moreover, if G has an effective hierarchy over X and for every non-linear \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with a non-trivial stabilizer, the group H_T has effective hierarchy over $Y(T)$ then H has an effective hierarchy over Y .

Proof. First of all, by Corollary 2 we can assume X to be finite. Hence, we can assume that any two distinct \mathbb{Z}^{n-1} -subtrees S and T of \mathcal{K} are labeled distinct alphabets $X(S)$ and $X(T)$. Next, by Lemma 8, in each \mathbb{Z}^{n-1} -subtree S of \mathcal{K} with trivial stabilizer there are only finitely many branch-points, so we can cut S along these branch-points, obtain finitely many closed and half-open segments, and relabel them by a finite alphabet. Thus we can assume all this to be done already.

Let T be a non-linear \mathbb{Z}^{n-1} -subtree of \mathcal{K} with a non-trivial stabilizer. Observe that by Lemma 9 every end a of T either is an end of Γ_{K_T} , or $a = g \cdot b$, where b is from a finite list of representatives of orbits of ends of $T - \Gamma_{K_T}$.

By the assumption, T embeds into T' labeled by $Y(T)$, while Γ_{K_T} embeds into $\Gamma_{H(T)}$, where $H(T)$ acts regularly on T' . It follows that for every branch-point b of T the label of $\psi_T([x_T, b])$ defines an element of $H(T)$. In particular,

the label of $\psi_T(d)$ belongs to $H(T)$ for every $d \in \mathcal{D}(T)$. Moreover, if S_1, S_2 are \mathbb{Z}^{n-1} -subtrees of \mathcal{K} adjacent to T and a_{S_1}, a_{S_2} are the corresponding ends of T then a_{S_1} is not $H(T)$ -equivalent to a_{S_2} . So, by the assumption, a_{S_1} is not $H(T)$ -equivalent to a_{S_2} and it follows that

$$(h_1 \cdot \theta([x_T, x_{S_1}] \cap T)) \cap (h_2 \cdot \theta([x_T, x_{S_2}] \cap T))$$

is a closed segment of T' , hence,

$$\text{com}(h_1 * c_{x_T, x_{S_1}}, h_2 * c_{x_T, x_{S_2}}),$$

is defined in $CDR(\mathbb{Z}^{n-1}, Y(T))$. Since $X(T) \cap X(S) = \emptyset$ then $h * c_{x_T, x_S}^{-1} = h \circ c_{x_T, x_S}^{-1}$ for every \mathbb{Z}^{n-1} -subtree S of \mathcal{K} adjacent to T . Thus,

$$\{H(T), \mathcal{D}(T), \{c_{x_T, x_S} \mid S \text{ is adjacent to } T \text{ in } \mathcal{K}\}\},$$

which is finite, generates a subgroup $H'(T)$ in $CDR(\mathbb{Z}^n, Q)$, where

$$Q = \bigcup_{T \in \mathcal{K}} Y(T),$$

so that T embeds into $\Gamma_{H'(T)}$. Moreover, $H'(T)$ acts regularly on $\Gamma_{H'(T)}$.

Now, from the fact that alphabet $X(T)$ is disjoint from $X(S)$ if T is not G -equivalent to S it follows that $\{H'(T) \mid T \in \mathcal{K}\}$ generates a subgroup H of $CDR(\mathbb{Z}^n, Y)$, where Y is a finite alphabet containing Q . Observe that $\Gamma_{H'(T)}$ embeds into Γ_H for each $T \in \mathcal{K}$. Moreover, for every $f, g \in H$ we have $w = Y(\varepsilon_H, f \cdot \varepsilon_H, g \cdot \varepsilon_H)$ belongs to one of the subtrees $\Gamma_{H'(T)}$, hence $[\varepsilon_H, w]$ defines an element of $H'(T) \subset H$. That is, H acts regularly on Γ_H .

Next, since

$$G \leq \langle K_T, \{\mathcal{D}(T) \mid T \in \mathcal{K}\} \rangle \leq H$$

then G embeds into H .

Finally, every end a of Γ_G uniquely corresponds to an end in Δ_G . Every end of Δ_G can be viewed as a reduced infinite path p_a in Δ_G originating at $v \in \Delta_G$ which is the image of $\varepsilon \in \Gamma_G$. Observe that two ends a and b of Γ_G are G -equivalent if and only if $\pi(p_a) = \pi(p_b)$ in Ψ_G .

Denote $\Delta_H = \Gamma_H / \sim$, where “ \sim ” is the equivalence of \mathbb{Z}^{n-1} -close points. Since $\psi : \Gamma_G \rightarrow \Gamma_H$ is an embedding then Δ_G embeds into Δ_H and with an abuse of notation we are going to denote this embedding by ψ . Let $w = \psi(v)$.

Let a and b be non- G -equivalent ends of Γ_G and let

$$p_a = v \ v_1 \ v_2 \ \cdots, \quad p_b = v \ u_1 \ u_2 \ \cdots.$$

Assume that $\psi(a)$ and $\psi(b)$ are H -equivalent in Γ_H , that is, there exists $h \in H$ such that $h \cdot p_{\psi(a)} = p_{\psi(b)}$. Since $p_{\psi(a)}$ and $p_{\psi(b)}$ have the same origin w then $h \cdot w = w$, that is, $h \in \text{Stab}_H(T'_0)$, where T'_0 is a \mathbb{Z}^{n-1} -subtree of Γ_H containing $\psi(T_0)$. Moreover, if $e_1 = (w, \psi(v_1))$, $f_1 = (w, \psi(u_1))$ then $h \cdot e_1 = f_1$ and it follows that $h \cdot a_1 = b_1$, where a_1 and b_1 are ends of $\psi(T_0)$ corresponding to e_1

and f_1 . By the assumption of the theorem there exists $\phi(g_1) \in \text{Stab}_{\phi(G)}(\psi(T_0))$ such that $\phi(g_1) \cdot a_1 = b_1$, so, $\phi(g_1) \cdot \psi(v_1) = \psi(u_1)$. Since $\phi : G \rightarrow H$ and $\psi : \Gamma_G \rightarrow \Gamma_H$ are embeddings, it follows that $g_1 \cdot v_1 = u_1$ and the images of $\pi(u_1) = \pi(v_1)$ in Δ_G .

Continuing in the same way we obtain $\pi(u_i) = \pi(v_i)$, $i \geq 1$ in Δ_G , so, a and b are G -equivalent which gives a contradiction with the assumption that $\psi(a)$ and $\psi(b)$ are H -equivalent in Γ_H .

The effectiveness part follows from the effectiveness parts of Lemma 8 and Lemma 9. □

Theorem 4. *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$, where X is arbitrary. Then there exists a finite alphabet Y and an embedding $\phi : G \rightarrow H$, where H is a finitely generated subgroup of $CDR(\mathbb{Z}^n, Y)$ with a regular length function, such that $|g|_G = |\phi(g)|_H$ for every $g \in G$. Moreover, if G has effective an hierarchy over X then H an effective hierarchy over Y .*

Proof. We use the induction on n . If $n = 1$ then the result follows from Lemma 6. Finally, the induction step follows from Theorem 3. □

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